

9.7 Taylor Polynomials and Approximations

- Find polynomial approximations of elementary functions and compare them with the elementary functions.
- Find Taylor and Maclaurin polynomial approximations of elementary functions.
- Use the remainder of a Taylor polynomial.

Polynomial Approximations of Elementary Functions

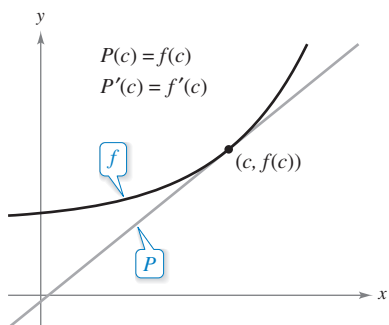
The goal of this section is to show how polynomial functions can be used as approximations for other elementary functions. To find a polynomial function P that approximates another function f , begin by choosing a number c in the domain of f at which f and P have the same value. That is,

$$P(c) = f(c) \quad \text{Graphs of } f \text{ and } P \text{ pass through } (c, f(c)).$$

The approximating polynomial is said to be **expanded about c** or **centered at c** . Geometrically, the requirement that $P(c) = f(c)$ means that the graph of P passes through the point $(c, f(c))$. Of course, there are many polynomials whose graphs pass through the point $(c, f(c))$. Your task is to find a polynomial whose graph resembles the graph of f near this point. One way to do this is to impose the additional requirement that the slope of the polynomial function be the same as the slope of the graph of f at the point $(c, f(c))$.

$$P'(c) = f'(c) \quad \text{Graphs of } f \text{ and } P \text{ have the same slope at } (c, f(c)).$$

With these two requirements, you can obtain a simple linear approximation of f , as shown in Figure 9.10.



Near $(c, f(c))$, the graph of P can be used to approximate the graph of f .

Figure 9.10



EXAMPLE 1 First-Degree Polynomial Approximation of $f(x) = e^x$

REMARK Example 1 is not the first time you have used a linear function to approximate another function. The same procedure was used as the basis for Newton's Method.

For the function $f(x) = e^x$, find a first-degree polynomial function $P_1(x) = a_0 + a_1x$ whose value and slope agree with the value and slope of f at $x = 0$.

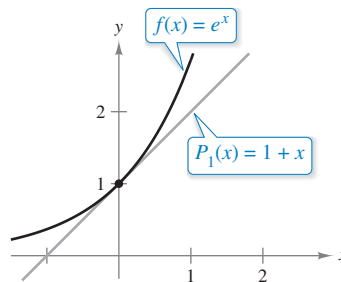
Solution Because $f(x) = e^x$ and $f'(x) = e^x$, the value and the slope of f at $x = 0$ are

$$f(0) = e^0 = 1 \quad \text{Value of } f \text{ at } x = 0$$

and

$$f'(0) = e^0 = 1. \quad \text{Slope of } f \text{ at } x = 0$$

Because $P_1(x) = a_0 + a_1x$, you can use the condition that $P_1(0) = f(0)$ to conclude that $a_0 = 1$. Moreover, because $P_1'(x) = a_1$, you can use the condition that $P_1'(0) = f'(0)$ to conclude that $a_1 = 1$. Therefore, $P_1(x) = 1 + x$. Figure 9.11 shows the graphs of $P_1(x) = 1 + x$ and $f(x) = e^x$.



P_1 is the first-degree polynomial approximation of $f(x) = e^x$.

Figure 9.11

In Figure 9.12, you can see that, at points near (0, 1), the graph of the first-degree polynomial function

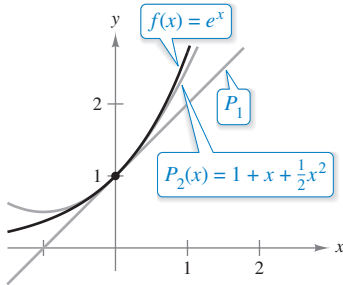
$$P_1(x) = 1 + x \quad \text{1st-degree approximation}$$

is reasonably close to the graph of $f(x) = e^x$. As you move away from (0, 1), however, the graphs move farther and farther from each other and the accuracy of the approximation decreases. To improve the approximation, you can impose yet another requirement—that the values of the second derivatives of P and f agree when $x = 0$. The polynomial, P_2 , of least degree that satisfies all three requirements $P_2(0) = f(0)$, $P_2'(0) = f'(0)$, and $P_2''(0) = f''(0)$ can be shown to be

$$P_2(x) = 1 + x + \frac{1}{2}x^2. \quad \text{2nd-degree approximation}$$

Moreover, in Figure 9.12, you can see that P_2 is a better approximation of f than P_1 . By requiring that the values of $P_n(x)$ and its first n derivatives match those of $f(x) = e^x$ at $x = 0$, you obtain the n th-degree approximation shown below.

$$P_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n \approx e^x \quad \text{nth-degree approximation}$$



P_2 is the second-degree polynomial approximation of $f(x) = e^x$.
Figure 9.12

EXAMPLE 2 Third-Degree Polynomial Approximation of $f(x) = e^x$

Construct a table comparing the values of the polynomial

$$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 \quad \text{3rd-degree approximation}$$

with $f(x) = e^x$ for several values of x near 0.

Solution Using a calculator or a computer, you can obtain the results shown in the table. Note that for $x = 0$, the two functions have the same value, but that as x moves farther away from 0, the accuracy of the approximating polynomial $P_3(x)$ decreases.

x	-1.0	-0.2	-0.1	0	0.1	0.2	1.0
e^x	0.3679	0.81873	0.904837	1	1.105171	1.22140	2.7183
$P_3(x)$	0.3333	0.81867	0.904833	1	1.105167	1.22133	2.6667

TECHNOLOGY A graphing utility can be used to compare the graph of the approximating polynomial with the graph of the function f . For instance, in Figure 9.13, the graph of

$$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \quad \text{3rd-degree approximation}$$

is compared with the graph of $f(x) = e^x$. If you have access to a graphing utility, try comparing the graphs of

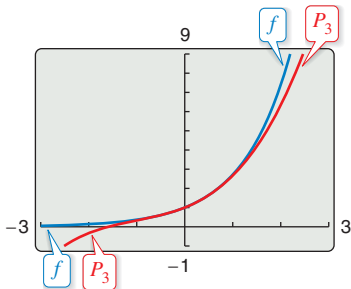
$$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 \quad \text{4th-degree approximation}$$

$$P_5(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 \quad \text{5th-degree approximation}$$

and

$$P_6(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 \quad \text{6th-degree approximation}$$

with the graph of f . What do you notice?



P_3 is the third-degree polynomial approximation of $f(x) = e^x$.
Figure 9.13



BROOK TAYLOR (1685–1731)

Although Taylor was not the first to seek polynomial approximations of transcendental functions, his account published in 1715 was one of the first comprehensive works on the subject. See *LarsonCalculus.com* to read more of this biography.

Taylor and Maclaurin Polynomials

The polynomial approximation of

$$f(x) = e^x$$

in Example 2 is expanded about $c = 0$. For expansions about an arbitrary value of c , it is convenient to write the polynomial in the form

$$P_n(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \cdots + a_n(x - c)^n.$$

In this form, repeated differentiation produces

$$P_n'(x) = a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + \cdots + na_n(x - c)^{n-1}$$

$$P_n''(x) = 2a_2 + 2(3a_3)(x - c) + \cdots + n(n - 1)a_n(x - c)^{n-2}$$

$$P_n'''(x) = 2(3a_3) + \cdots + n(n - 1)(n - 2)a_n(x - c)^{n-3}$$

⋮

$$P_n^{(n)}(x) = n(n - 1)(n - 2) \cdots (2)(1)a_n.$$

Letting $x = c$, you then obtain

$$P_n(c) = a_0, \quad P_n'(c) = a_1, \quad P_n''(c) = 2a_2, \dots, \quad P_n^{(n)}(c) = n!a_n$$

and because the values of f and its first n derivatives must agree with the values of P_n and its first n derivatives at $x = c$, it follows that

$$f(c) = a_0, \quad f'(c) = a_1, \quad \frac{f''(c)}{2!} = a_2, \quad \dots, \quad \frac{f^{(n)}(c)}{n!} = a_n.$$

With these coefficients, you can obtain the following definition of **Taylor polynomials**, named after the English mathematician Brook Taylor, and **Maclaurin polynomials**, named after the English mathematician Colin Maclaurin (1698–1746).

Definitions of n th Taylor Polynomial and n th Maclaurin Polynomial

If f has n derivatives at c , then the polynomial

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

is called the **n th Taylor polynomial for f at c** . If $c = 0$, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

is also called the **n th Maclaurin polynomial for f** .

•• **REMARK** Maclaurin polynomials are special types of Taylor polynomials for which $c = 0$.

EXAMPLE 3 A Maclaurin Polynomial for $f(x) = e^x$

Find the n th Maclaurin polynomial for

$$f(x) = e^x.$$

Solution From the discussion on the preceding page, the n th Maclaurin polynomial for

$$f(x) = e^x$$

is

$$P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n.$$

FOR FURTHER INFORMATION

To see how to use series to obtain other approximations to e , see the article “Novel Series-based Approximations to e ” by John Knox and Harlan J. Brothers in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

EXAMPLE 4 Finding Taylor Polynomials for $\ln x$

Find the Taylor polynomials P_0 , P_1 , P_2 , P_3 , and P_4 for

$$f(x) = \ln x$$

centered at $c = 1$.

Solution Expanding about $c = 1$ yields the following.

$$\begin{aligned} f(x) &= \ln x & f(1) &= \ln 1 = 0 \\ f'(x) &= \frac{1}{x} & f'(1) &= \frac{1}{1} = 1 \\ f''(x) &= -\frac{1}{x^2} & f''(1) &= -\frac{1}{1^2} = -1 \\ f'''(x) &= \frac{2!}{x^3} & f'''(1) &= \frac{2!}{1^3} = 2 \\ f^{(4)}(x) &= -\frac{3!}{x^4} & f^{(4)}(1) &= -\frac{3!}{1^4} = -6 \end{aligned}$$

Therefore, the Taylor polynomials are as follows.

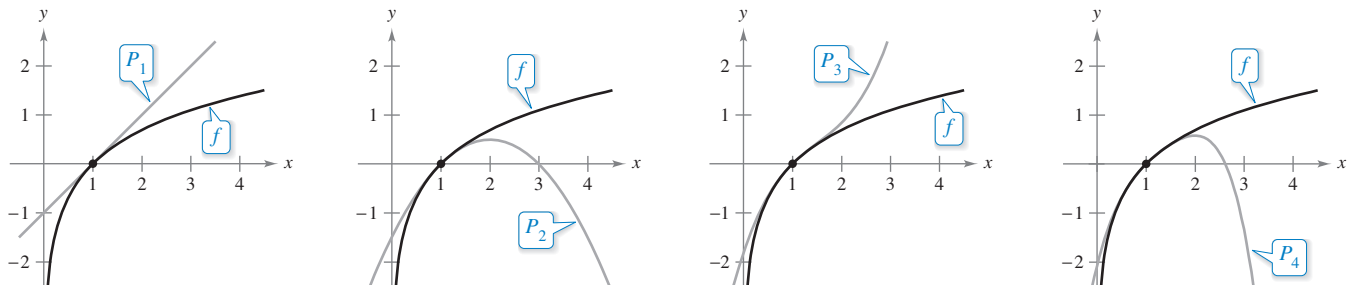
$$\begin{aligned} P_0(x) &= f(1) = 0 \\ P_1(x) &= f(1) + f'(1)(x - 1) = (x - 1) \\ P_2(x) &= f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 \\ &= (x - 1) - \frac{1}{2}(x - 1)^2 \\ P_3(x) &= f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 + \frac{f'''(1)}{3!}(x - 1)^3 \\ &= (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 \\ P_4(x) &= f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 + \frac{f'''(1)}{3!}(x - 1)^3 + \frac{f^{(4)}(1)}{4!}(x - 1)^4 \\ &= (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4 \end{aligned}$$

Figure 9.14 compares the graphs of P_1 , P_2 , P_3 , and P_4 with the graph of $f(x) = \ln x$. Note that near $x = 1$, the graphs are nearly indistinguishable. For instance,

$$P_4(1.1) \approx 0.0953083$$

and

$$\ln(1.1) \approx 0.0953102.$$



As n increases, the graph of P_n becomes a better and better approximation of the graph of $f(x) = \ln x$ near $x = 1$.

Figure 9.14

EXAMPLE 5 Finding Maclaurin Polynomials for $\cos x$

Find the Maclaurin polynomials P_0 , P_2 , P_4 , and P_6 for $f(x) = \cos x$. Use $P_6(x)$ to approximate the value of $\cos(0.1)$.

Solution Expanding about $c = 0$ yields the following.

$$\begin{aligned} f(x) &= \cos x & f(0) &= \cos 0 = 1 \\ f'(x) &= -\sin x & f'(0) &= -\sin 0 = 0 \\ f''(x) &= -\cos x & f''(0) &= -\cos 0 = -1 \\ f'''(x) &= \sin x & f'''(0) &= \sin 0 = 0 \end{aligned}$$

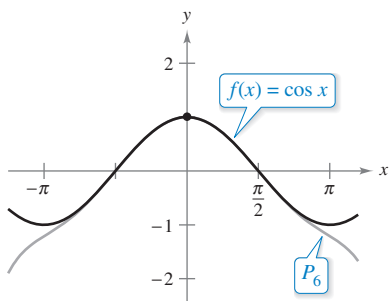
Through repeated differentiation, you can see that the pattern 1, 0, -1, 0 continues, and you obtain the Maclaurin polynomials

$$P_0(x) = 1, \quad P_2(x) = 1 - \frac{1}{2!}x^2, \quad P_4(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4,$$

and

$$P_6(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6.$$

Using $P_6(x)$, you obtain the approximation $\cos(0.1) \approx 0.995004165$, which coincides with the calculator value to nine decimal places. Figure 9.15 compares the graphs of $f(x) = \cos x$ and P_6 .



Near $(0, 1)$, the graph of P_6 can be used to approximate the graph of $f(x) = \cos x$.

Figure 9.15

Note in Example 5 that the Maclaurin polynomials for $\cos x$ have only even powers of x . Similarly, the Maclaurin polynomials for $\sin x$ have only odd powers of x (see Exercise 17). This is not generally true of the Taylor polynomials for $\sin x$ and $\cos x$ expanded about $c \neq 0$, as you can see in the next example.

EXAMPLE 6 Finding a Taylor Polynomial for $\sin x$

•••▶ See LarsonCalculus.com for an interactive version of this type of example.

Find the third Taylor polynomial for $f(x) = \sin x$, expanded about $c = \pi/6$.

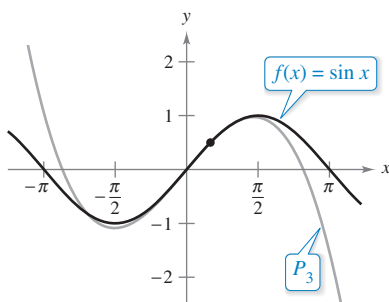
Solution Expanding about $c = \pi/6$ yields the following.

$$\begin{aligned} f(x) &= \sin x & f\left(\frac{\pi}{6}\right) &= \sin \frac{\pi}{6} = \frac{1}{2} \\ f'(x) &= \cos x & f'\left(\frac{\pi}{6}\right) &= \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \\ f''(x) &= -\sin x & f''\left(\frac{\pi}{6}\right) &= -\sin \frac{\pi}{6} = -\frac{1}{2} \\ f'''(x) &= -\cos x & f'''\left(\frac{\pi}{6}\right) &= -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2} \end{aligned}$$

So, the third Taylor polynomial for $f(x) = \sin x$, expanded about $c = \pi/6$, is

$$\begin{aligned} P_3(x) &= f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) + \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{f'''\left(\frac{\pi}{6}\right)}{3!}\left(x - \frac{\pi}{6}\right)^3 \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{2(2!)}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{2(3!)}\left(x - \frac{\pi}{6}\right)^3. \end{aligned}$$

Figure 9.16 compares the graphs of $f(x) = \sin x$ and P_3 .



Near $(\pi/6, 1/2)$, the graph of P_3 can be used to approximate the graph of $f(x) = \sin x$.

Figure 9.16

Taylor polynomials and Maclaurin polynomials can be used to approximate the value of a function at a specific point. For instance, to approximate the value of $\ln(1.1)$, you can use Taylor polynomials for $f(x) = \ln x$ expanded about $c = 1$, as shown in Example 4, or you can use Maclaurin polynomials, as shown in Example 7.

EXAMPLE 7 Approximation Using Maclaurin Polynomials

Use a fourth Maclaurin polynomial to approximate the value of $\ln(1.1)$.

Solution Because 1.1 is closer to 1 than to 0, you should consider Maclaurin polynomials for the function $g(x) = \ln(1 + x)$.

$$\begin{aligned} g(x) &= \ln(1 + x) & g(0) &= \ln(1 + 0) = 0 \\ g'(x) &= (1 + x)^{-1} & g'(0) &= (1 + 0)^{-1} = 1 \\ g''(x) &= -(1 + x)^{-2} & g''(0) &= -(1 + 0)^{-2} = -1 \\ g'''(x) &= 2(1 + x)^{-3} & g'''(0) &= 2(1 + 0)^{-3} = 2 \\ g^{(4)}(x) &= -6(1 + x)^{-4} & g^{(4)}(0) &= -6(1 + 0)^{-4} = -6 \end{aligned}$$

Exploration
Check to see that the fourth Taylor polynomial (from Example 4), evaluated at $x = 1.1$, yields the same result as the fourth Maclaurin polynomial in Example 7.

Note that you obtain the same coefficients as in Example 4. Therefore, the fourth Maclaurin polynomial for $g(x) = \ln(1 + x)$ is

$$\begin{aligned} P_4(x) &= g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 + \frac{g^{(4)}(0)}{4!}x^4 \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4. \end{aligned}$$

Consequently,

$$\ln(1.1) = \ln(1 + 0.1) \approx P_4(0.1) \approx 0.0953083.$$

The table below illustrates the accuracy of the Maclaurin polynomial approximation of the calculator value of $\ln(1.1)$. You can see that as n increases, $P_n(0.1)$ approaches the calculator value of 0.0953102.

Maclaurin Polynomials and Approximations of $\ln(1 + x)$ at $x = 0.1$

n	1	2	3	4
$P_n(0.1)$	0.1000000	0.0950000	0.0953333	0.0953083

On the other hand, the table below illustrates that as you move away from the expansion point $c = 0$, the accuracy of the approximation decreases.

Fourth Maclaurin Polynomial Approximation of $\ln(1 + x)$

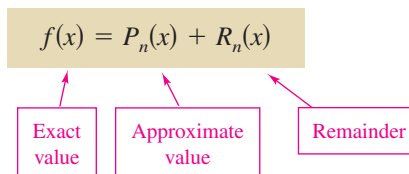
x	0	0.1	0.5	0.75	1.0
$\ln(1 + x)$	0	0.0953102	0.4054651	0.5596158	0.6931472
$P_4(x)$	0	0.0953083	0.4010417	0.5302734	0.5833333

These two tables illustrate two very important points about the accuracy of Taylor (or Maclaurin) polynomials for use in approximations.

1. The approximation is usually better for higher-degree Taylor (or Maclaurin) polynomials than for those of lower degree.
2. The approximation is usually better at x -values close to c than at x -values far from c .

Remainder of a Taylor Polynomial

An approximation technique is of little value without some idea of its accuracy. To measure the accuracy of approximating a function value $f(x)$ by the Taylor polynomial $P_n(x)$, you can use the concept of a **remainder** $R_n(x)$, defined as follows.



So, $R_n(x) = f(x) - P_n(x)$. The absolute value of $R_n(x)$ is called the **error** associated with the approximation. That is,

$$\text{Error} = |R_n(x)| = |f(x) - P_n(x)|.$$

The next theorem gives a general procedure for estimating the remainder associated with a Taylor polynomial. This important theorem is called **Taylor's Theorem**, and the remainder given in the theorem is called the **Lagrange form of the remainder**.

THEOREM 9.19 Taylor's Theorem

If a function f is differentiable through order $n + 1$ in an interval I containing c , then, for each x in I , there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - c)^{n+1}.$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

One useful consequence of Taylor's Theorem is that

$$|R_n(x)| \leq \frac{|x - c|^{n+1}}{(n+1)!} \max |f^{(n+1)}(z)|$$

where $\max |f^{(n+1)}(z)|$ is the maximum value of $f^{(n+1)}(z)$ between x and c .

For $n = 0$, Taylor's Theorem states that if f is differentiable in an interval I containing c , then, for each x in I , there exists z between x and c such that

$$f(x) = f(c) + f'(z)(x - c) \quad \text{or} \quad f'(z) = \frac{f(x) - f(c)}{x - c}.$$

Do you recognize this special case of Taylor's Theorem? (It is the Mean Value Theorem.)

When applying Taylor's Theorem, you should not expect to be able to find the exact value of z . (If you could do this, an approximation would not be necessary.) Rather, you are trying to find bounds for $f^{(n+1)}(z)$ from which you are able to tell how large the remainder $R_n(x)$ is.

EXAMPLE 8 Determining the Accuracy of an Approximation

The third Maclaurin polynomial for $\sin x$ is

$$P_3(x) = x - \frac{x^3}{3!}$$

Use Taylor's Theorem to approximate $\sin(0.1)$ by $P_3(0.1)$ and determine the accuracy of the approximation.

Solution Using Taylor's Theorem, you have

$$\sin x = x - \frac{x^3}{3!} + R_3(x) = x - \frac{x^3}{3!} + \frac{f^{(4)}(z)}{4!} x^4$$

where $0 < z < 0.1$. Therefore,

$$\sin(0.1) \approx 0.1 - \frac{(0.1)^3}{3!} \approx 0.1 - 0.000167 = 0.099833.$$

Because $f^{(4)}(z) = \sin z$, it follows that the error $|R_3(0.1)|$ can be bounded as follows.

$$0 < R_3(0.1) = \frac{\sin z}{4!} (0.1)^4 < \frac{0.0001}{4!} \approx 0.000004$$

This implies that

$$0.099833 < \sin(0.1) \approx 0.099833 + R_3(0.1) < 0.099833 + 0.000004$$

or

$$0.099833 < \sin(0.1) < 0.099837.$$

•• **REMARK** Note that when you use a calculator,
 $\sin(0.1) \approx 0.0998334.$



EXAMPLE 9 Approximating a Value to a Desired Accuracy

Determine the degree of the Taylor polynomial $P_n(x)$ expanded about $c = 1$ that should be used to approximate $\ln(1.2)$ so that the error is less than 0.001.

Solution Following the pattern of Example 4, you can see that the $(n + 1)$ st derivative of $f(x) = \ln x$ is

$$f^{(n+1)}(x) = (-1)^n \frac{n!}{x^{n+1}}$$

Using Taylor's Theorem, you know that the error $|R_n(1.2)|$ is

$$\begin{aligned} |R_n(1.2)| &= \left| \frac{f^{(n+1)}(z)}{(n+1)!} (1.2 - 1)^{n+1} \right| \\ &= \frac{n!}{z^{n+1}} \left[\frac{1}{(n+1)!} \right] (0.2)^{n+1} \\ &= \frac{(0.2)^{n+1}}{z^{n+1}(n+1)} \end{aligned}$$

where $1 < z < 1.2$. In this interval, $(0.2)^{n+1}/[z^{n+1}(n+1)]$ is less than $(0.2)^{n+1}/(n+1)$. So, you are seeking a value of n such that

$$\frac{(0.2)^{n+1}}{(n+1)} < 0.001 \implies 1000 < (n+1)5^{n+1}.$$

By trial and error, you can determine that the least value of n that satisfies this inequality is $n = 3$. So, you would need the third Taylor polynomial to achieve the desired accuracy in approximating $\ln(1.2)$.

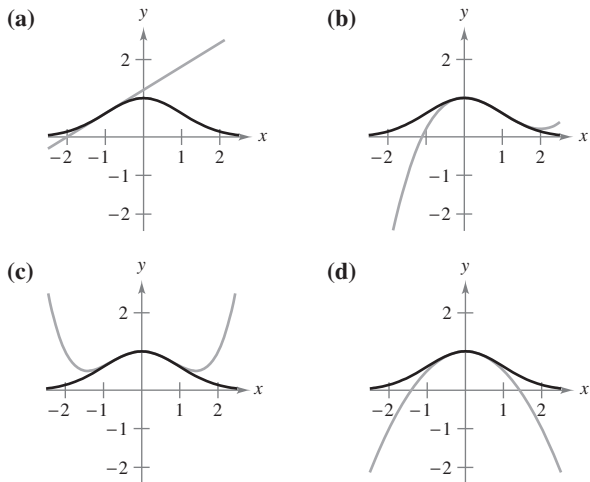
•• **REMARK** Note that when you use a calculator,
 $P_3(1.2) \approx 0.1827$
 and
 $\ln(1.2) \approx 0.1823.$



9.7 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Matching In Exercises 1–4, match the Taylor polynomial approximation of the function $f(x) = e^{-x^2/2}$ with the corresponding graph. [The graphs are labeled (a), (b), (c), and (d).]



1. $g(x) = -\frac{1}{2}x^2 + 1$
2. $g(x) = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$
3. $g(x) = e^{-1/2}[(x + 1) + 1]$
4. $g(x) = e^{-1/2}[\frac{1}{3}(x - 1)^3 - (x - 1) + 1]$

Graphical and Numerical Analysis In Exercises 5–8, find a first-degree polynomial function P_1 whose value and slope agree with the value and slope of f at $x = c$. Use a graphing utility to graph f and P_1 . What is P_1 called?

5. $f(x) = \frac{\sqrt{x}}{4}, c = 4$
6. $f(x) = \frac{6}{\sqrt[3]{x}}, c = 8$
7. $f(x) = \sec x, c = \frac{\pi}{4}$
8. $f(x) = \tan x, c = \frac{\pi}{4}$

Graphical and Numerical Analysis In Exercises 9 and 10, use a graphing utility to graph f and its second-degree polynomial approximation P_2 at $x = c$. Complete the table comparing the values of f and P_2 .

9. $f(x) = \frac{4}{\sqrt{x}}, c = 1$
 $P_2(x) = 4 - 2(x - 1) + \frac{3}{2}(x - 1)^2$

x	0	0.8	0.9	1	1.1	1.2	2
$f(x)$							
$P_2(x)$							

10. $f(x) = \sec x, c = \frac{\pi}{4}$

$$P_2(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3}{2}\sqrt{2}\left(x - \frac{\pi}{4}\right)^2$$

x	-2.15	0.585	0.685	$\frac{\pi}{4}$	0.885	0.985	1.785
$f(x)$							
$P_2(x)$							

Conjecture Consider the function $f(x) = \cos x$ and its Maclaurin polynomials $P_2, P_4,$ and P_6 (see Example 5).

- (a) Use a graphing utility to graph f and the indicated polynomial approximations.
- (b) Evaluate and compare the values of $f^{(n)}(0)$ and $P_n^{(n)}(0)$ for $n = 2, 4,$ and 6 .
- (c) Use the results in part (b) to make a conjecture about $f^{(n)}(0)$ and $P_n^{(n)}(0)$.

Conjecture Consider the function $f(x) = x^2e^x$.

- (a) Find the Maclaurin polynomials $P_2, P_3,$ and P_4 for f .
- (b) Use a graphing utility to graph $f, P_2, P_3,$ and P_4 .
- (c) Evaluate and compare the values of $f^{(n)}(0)$ and $P_n^{(n)}(0)$ for $n = 2, 3,$ and 4 .
- (d) Use the results in part (c) to make a conjecture about $f^{(n)}(0)$ and $P_n^{(n)}(0)$.

Finding a Maclaurin Polynomial In Exercises 13–24, find the n th Maclaurin polynomial for the function.

13. $f(x) = e^{4x}, n = 4$
14. $f(x) = e^{-x}, n = 5$
15. $f(x) = e^{-x/2}, n = 4$
16. $f(x) = e^{x/3}, n = 4$
17. $f(x) = \sin x, n = 5$
18. $f(x) = \cos \pi x, n = 4$
19. $f(x) = xe^x, n = 4$
20. $f(x) = x^2e^{-x}, n = 4$
21. $f(x) = \frac{1}{x+1}, n = 5$
22. $f(x) = \frac{x}{x+1}, n = 4$
23. $f(x) = \sec x, n = 2$
24. $f(x) = \tan x, n = 3$

Finding a Taylor Polynomial In Exercises 25–30, find the n th Taylor polynomial centered at c .

25. $f(x) = \frac{2}{x}, n = 3, c = 1$
26. $f(x) = \frac{1}{x^2}, n = 4, c = 2$
27. $f(x) = \sqrt{x}, n = 3, c = 4$
28. $f(x) = \sqrt[3]{x}, n = 3, c = 8$
29. $f(x) = \ln x, n = 4, c = 2$
30. $f(x) = x^2 \cos x, n = 2, c = \pi$

CA **Finding Taylor Polynomials Using Technology** In Exercises 31 and 32, use a computer algebra system to find the indicated Taylor polynomials for the function f . Graph the function and the Taylor polynomials.

31. $f(x) = \tan \pi x$ 32. $f(x) = \frac{1}{x^2 + 1}$
 (a) $n = 3, c = 0$ (a) $n = 4, c = 0$
 (b) $n = 3, c = 1/4$ (b) $n = 4, c = 1$

33. Numerical and Graphical Approximations

(a) Use the Maclaurin polynomials $P_1(x)$, $P_3(x)$, and $P_5(x)$ for $f(x) = \sin x$ to complete the table.

x	0	0.25	0.50	0.75	1.00
$\sin x$	0	0.2474	0.4794	0.6816	0.8415
$P_1(x)$					
$P_3(x)$					
$P_5(x)$					

- CA** (b) Use a graphing utility to graph $f(x) = \sin x$ and the Maclaurin polynomials in part (a).
 (c) Describe the change in accuracy of a polynomial approximation as the distance from the point where the polynomial is centered increases.

34. Numerical and Graphical Approximations

(a) Use the Taylor polynomials $P_1(x)$, $P_2(x)$, and $P_4(x)$ for $f(x) = e^x$ centered at $c = 1$ to complete the table.

x	1.00	1.25	1.50	1.75	2.00
e^x	e	3.4903	4.4817	5.7546	7.3891
$P_1(x)$					
$P_2(x)$					
$P_4(x)$					

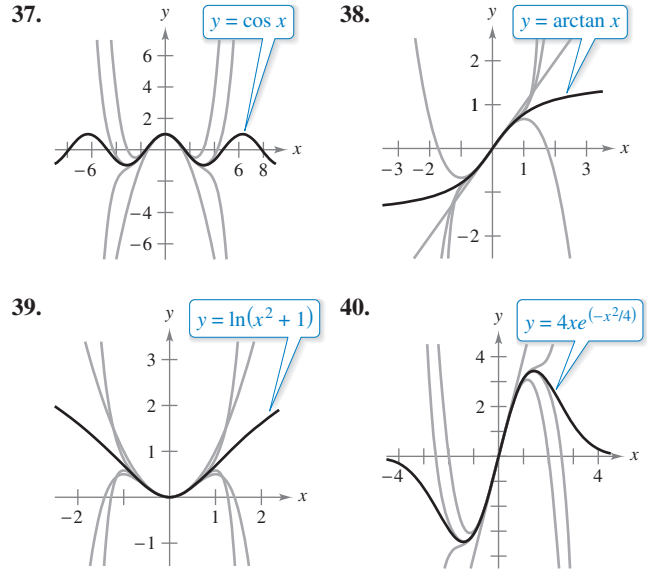
- CA** (b) Use a graphing utility to graph $f(x) = e^x$ and the Taylor polynomials in part (a).
 (c) Describe the change in accuracy of polynomial approximations as the degree increases.

Numerical and Graphical Approximations In Exercises 35 and 36, (a) find the Maclaurin polynomial $P_3(x)$ for $f(x)$, (b) complete the table for $f(x)$ and $P_3(x)$, and (c) sketch the graphs of $f(x)$ and $P_3(x)$ on the same set of coordinate axes.

x	-0.75	-0.50	-0.25	0	0.25	0.50	0.75
$f(x)$							
$P_3(x)$							

35. $f(x) = \arcsin x$ 36. $f(x) = \arctan x$

Identifying Maclaurin Polynomials In Exercises 37–40, the graph of $y = f(x)$ is shown with four of its Maclaurin polynomials. Identify the Maclaurin polynomials and use a graphing utility to confirm your results.



Approximating a Function Value In Exercises 41–44, approximate the function at the given value of x , using the polynomial found in the indicated exercise.


41. $f(x) = e^{4x}, f\left(\frac{1}{4}\right)$, Exercise 13
 42. $f(x) = x^2e^{-x}, f\left(\frac{1}{5}\right)$, Exercise 20
 43. $f(x) = \ln x, f(2.1)$, Exercise 29
 44. $f(x) = x^2 \cos x, f\left(\frac{7\pi}{8}\right)$, Exercise 30

Using Taylor's Theorem In Exercises 45–48, use Taylor's Theorem to obtain an upper bound for the error of the approximation. Then calculate the exact value of the error.

45. $\cos(0.3) \approx 1 - \frac{(0.3)^2}{2!} + \frac{(0.3)^4}{4!}$
 46. $e \approx 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} + \frac{1^5}{5!}$
 47. $\arcsin(0.4) \approx 0.4 + \frac{(0.4)^3}{2 \cdot 3}$
 48. $\arctan(0.4) \approx 0.4 - \frac{(0.4)^3}{3}$

Finding a Degree In Exercises 49–52, determine the degree of the Maclaurin polynomial required for the error in the approximation of the function at the indicated value of x to be less than 0.001.

49. $\sin(0.3)$
 50. $\cos(0.1)$
 51. $e^{0.6}$
 52. $\ln(1.25)$

 **Finding a Degree Using Technology** In Exercises 53 and 54, determine the degree of the Maclaurin polynomial required for the error in the approximation of the function at the indicated value of x to be less than 0.0001. Use a computer algebra system to obtain and evaluate the required derivative.

53. $f(x) = \ln(x + 1)$, approximate $f(0.5)$.

54. $f(x) = e^{-\pi x}$, approximate $f(1.3)$.

Finding Values In Exercises 55–58, determine the values of x for which the function can be replaced by the Taylor polynomial if the error cannot exceed 0.001.

55. $f(x) = e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$, $x < 0$

56. $f(x) = \sin x \approx x - \frac{x^3}{3!}$

57. $f(x) = \cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

58. $f(x) = e^{-2x} \approx 1 - 2x + 2x^2 - \frac{4}{3}x^3$

WRITING ABOUT CONCEPTS

59. Polynomial Approximation An elementary function is approximated by a polynomial. In your own words, describe what is meant by saying that the polynomial is *expanded about c* or *centered at c* .

60. Polynomial Approximation When an elementary function f is approximated by a second-degree polynomial P_2 centered at c , what is known about f and P_2 at c ? Explain your reasoning.

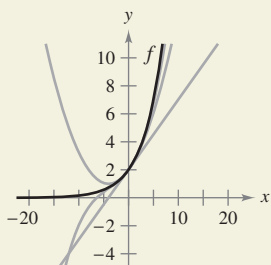
61. Taylor Polynomial State the definition of an n th-degree Taylor polynomial of f centered at c .

62. Accuracy of a Taylor Polynomial Describe the accuracy of the n th-degree Taylor polynomial of f centered at c as the distance between c and x increases.

63. Accuracy of a Taylor Polynomial In general, how does the accuracy of a Taylor polynomial change as the degree of the polynomial increases? Explain your reasoning.



64. HOW DO YOU SEE IT? The graphs show first-, second-, and third-degree polynomial approximations P_1 , P_2 , and P_3 of a function f . Label the graphs of P_1 , P_2 , and P_3 . To print an enlarged copy of the graph, go to MathGraphs.com.



65. Comparing Maclaurin Polynomials

(a) Compare the Maclaurin polynomials of degree 4 and degree 5, respectively, for the functions $f(x) = e^x$ and $g(x) = xe^x$. What is the relationship between them?

(b) Use the result in part (a) and the Maclaurin polynomial of degree 5 for $f(x) = \sin x$ to find a Maclaurin polynomial of degree 6 for the function $g(x) = x \sin x$.

(c) Use the result in part (a) and the Maclaurin polynomial of degree 5 for $f(x) = \sin x$ to find a Maclaurin polynomial of degree 4 for the function $g(x) = (\sin x)/x$.

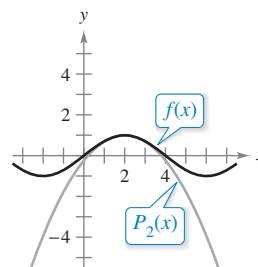
66. Differentiating Maclaurin Polynomials

(a) Differentiate the Maclaurin polynomial of degree 5 for $f(x) = \sin x$ and compare the result with the Maclaurin polynomial of degree 4 for $g(x) = \cos x$.

(b) Differentiate the Maclaurin polynomial of degree 6 for $f(x) = \cos x$ and compare the result with the Maclaurin polynomial of degree 5 for $g(x) = \sin x$.

(c) Differentiate the Maclaurin polynomial of degree 4 for $f(x) = e^x$. Describe the relationship between the two series.

67. Graphical Reasoning The figure shows the graphs of the function $f(x) = \sin(\pi x/4)$ and the second-degree Taylor polynomial $P_2(x) = 1 - (\pi^2/32)(x - 2)^2$ centered at $x = 2$.



(a) Use the symmetry of the graph of f to write the second-degree Taylor polynomial $Q_2(x)$ for f centered at $x = -2$.

(b) Use a horizontal translation of the result in part (a) to find the second-degree Taylor polynomial $R_2(x)$ for f centered at $x = 6$.

(c) Is it possible to use a horizontal translation of the result in part (a) to write a second-degree Taylor polynomial for f centered at $x = 4$? Explain.

68. Proof Prove that if f is an odd function, then its n th Maclaurin polynomial contains only terms with odd powers of x .

69. Proof Prove that if f is an even function, then its n th Maclaurin polynomial contains only terms with even powers of x .

70. Proof Let $P_n(x)$ be the n th Taylor polynomial for f at c . Prove that $P_n(c) = f(c)$ and $P^{(k)}(c) = f^{(k)}(c)$ for $1 \leq k \leq n$. (See Exercises 9 and 10.)

71. Writing The proof in Exercise 70 guarantees that the Taylor polynomial and its derivatives agree with the function and its derivatives at $x = c$. Use the graphs and tables in Exercises 33–36 to discuss what happens to the accuracy of the Taylor polynomial as you move away from $x = c$.